Factoring Quartics Which Have No Real Factors

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Teachers Note:

Quartics are a category of polynomial functions, called quartic because they involve fourth-degree exponents. The polynomial family is one of the basic categories of mathematical expressions, and its members are studied at every level of mathematics because of their power and simplicity. Perhaps most importantly for our students as they prepare for college level math, polynomials form an important bridge in calculus. They are easily examined using the tools of calculus but can also be used as building blocks to represent much more complicated expressions. For the past several hundred years, the factoring (writing in a form involving multiplication) of polynomials has been used as a tool to solve a host of important problems. The goal of this project was to analyze the factorizations of a particularly messy set of quartic polynomials.

Abstract: For Those with Little Time

My problem was to factor a quartic with only non-real linear factors. I found one real pair of quadratic factors and one imaginary pair of quadratic factors of the quartic, but they both had the same linear factors. I concluded that changing the order of the linear factors gives you different quadratic factors to the same quartic.

The Problem: A Wolf in Sheep's Clothing

I started with the problem of how to factor (x^n-a^n) and (x^n+a^n), and started to solve it, but in the process I found a very interesting quartic, which the rest of my investigation centered around:

$x^4+2x^3+4x^2+8x+16$

This quartic came from factoring (x-2) out of (x^5-2^5):

$(x-2)(x^5-2^5) = x^4+2x^3+4x^2+8x+16$

I wanted to factor this quartic further to get a full factorization. I graphed it to find the zeroes, but the graph looked like this:

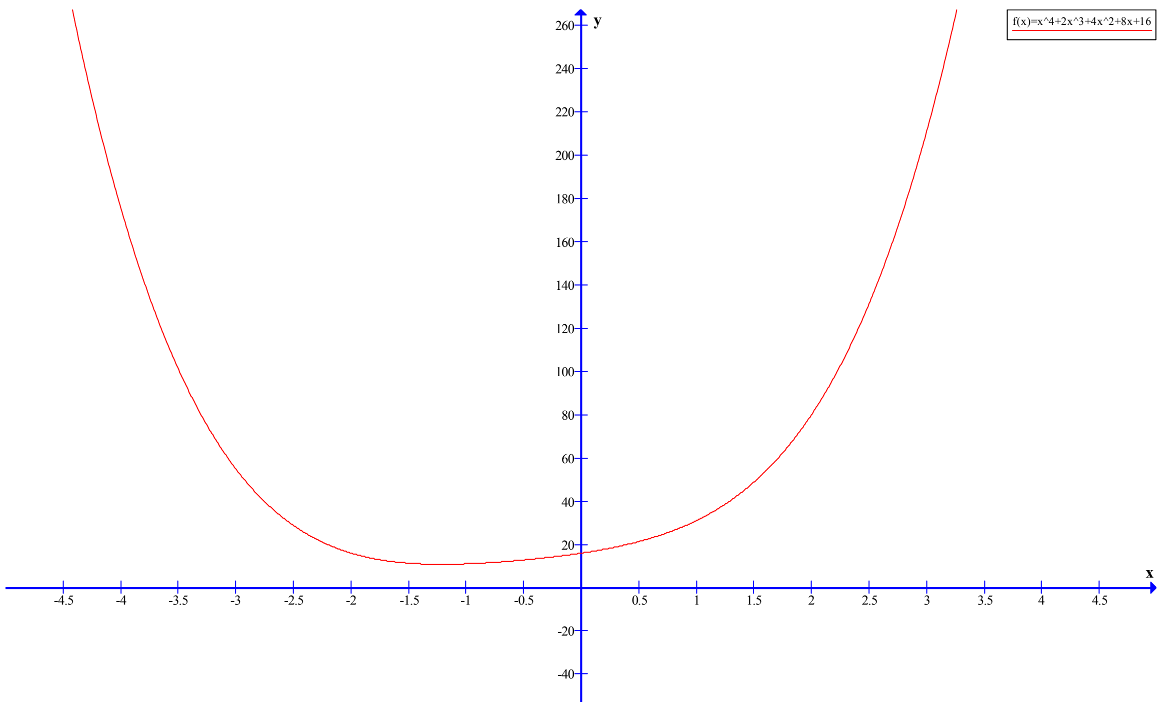

Figure 1

This meant that it had no real factors. But according to the fundamental theorem of algebra, this quartic must have 4 linear factors. So I knew that all four linear factors had to be complex. But how could I find them?
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First Steps: Creating a System of Equations

The way I decided to approach the problem was to factor the quartic into two quadratics and then use the quadratic formula to further factor these quadratics into four linear factors. This proved to be more complicated than I had expected. First, I set up a system of equations, which I would be able to solve to find the two quadratics. To start, I defined my quadratic factors as:

$$(x^2+ax+b)(x^2+cx+d)$$

If I could find a, b, c, and d, which are the coefficients, I would have my quadratics and be nearly finished with my problem. To make my system, I considered where the coefficients come from in my original quartic.

$$x^4+2x^3+4x^2+8x+16$$

When you expand the multiplication of the quadratic factors, it becomes:

$$x^4+ax^3+cx^3+bx^2+dx^2+acx^2+adx+bcx+bd$$

The first coefficient of the original quartic is 1, so it is irrelevant.

The second coefficient is 2, and it is attached to x^3. Therefore, whatever coefficients in my quadratics are attached to x's that will multiply to become x^3 must add to be 2. $2x^3$ must come from $x^2(cx) + x^2(ax)$. This gave me the equation $2x^3 = ax^3 + cx^3$. You can see this in the expansion of the quadratic multiplication. If I divide both sides by x^3, it becomes 2 = a+c. This was the first equation in my system.

The third coefficient in the quartic is 4. It is attached to x^2, so I had to find the parts of the quadratics that multiplied to become x^2. My equation was $4x^2 = ax(cx)+bx^2+dx^2$. When you divide both sides by x^2, it becomes 4 = ac+b+d. This was the second equation in my system.

My third equation came from the fourth quartic coefficient: 8. The equation that comes from this is 8x = adx+bcx, which when you divide out the x becomes 8 = ad+bc. This was the third equation in my system.

The final equation had to do with the fifth, and last, coefficient of the quartic, 16. I knew that 16 = bd, and no division was necessary. This was the final equation in my system.

So I finally had my system, which ended up as:

E1: 2 = a+c
E2: 4 = ac+b+d
E3: 8 = ad+bc
E4: 16 = bd

I had four equations and four variables, so I knew that I could most likely solve the system. The next step was to figure out how.
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Real Quadratics: Eye Opening Failure

Naturally, I assumed that my two quadratic factors would have real coefficients. Solving this system over the set of real numbers seemed simple enough, so I began using substitution to find an answer. I used substitution as well as the quadratic formula to solve the system, but no matter what I did, I could never satisfy all four equations. I came up with many different answers, but the most that I could satisfy at a time was three equations. As one final effort, I substituted out three of the variables and created a function that summarized my system with only one function:

$$y = \frac{16-4x}{-x+\frac{16}{x}} - \left(\frac{8-2x}{-x+\frac{16}{x}}\right)^2 + \frac{16}{x} + x - 4$$

I graphed this function and looked for the zeroes, because I had made the left side of the equation equal to 0. Sadly, the graph looked like this:

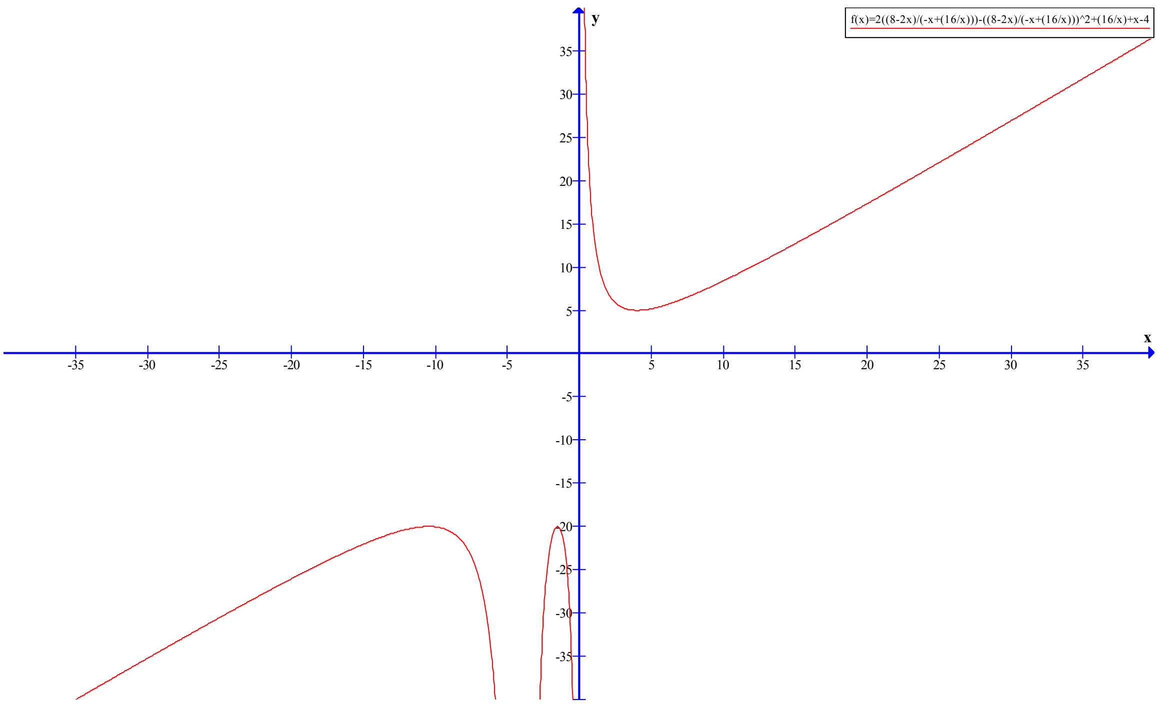

Figure 2

The function never crosses the x-axis, which means that it has no real solutions. I was thoroughly puzzled at this point, because I knew that the quartic had to have quadratic factors. It was then suggested to me that the coefficients of the quadratic factors may be complex. I had not even considered this option, and it sounded quite daunting.

Complex Quadratics: Another Brave Foray into the World of Failure

To start my experimenting with imaginary quadratics, I made a list of statements supported by my four equations to make the solution more manageable. The most important results that came out of this list were that a and c, as well as b and d, are conjugate pairs. This has to be true because the pairs need to multiply together to equal real numbers. This fact proved useful in later work, as I could now narrow down my problem to only two variables. I tried to manipulate my equations to give me a solution over the set of complex numbers, but I still could not satisfy all four equations. I could not figure out what was wrong, so I decided to go back and check my work.

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Mathematica: Revelations

I used Mathematica to check my arithmetic and algebra, and in the process I found two relations that I had used to turn my four variable system into a two variable system. These two equations were:

$$0 = -c^2 + 2c + \frac{16}{d} + d - 4$$

and

$$0 = (2-c)d + \frac{16c}{d} - 8$$

I used Mathematica to solve this system of two equations and two variables, and it gave me these answers to the system:

c = $\frac{1}{4}(4 - i\sqrt{50-10\sqrt{5}} - 3i\sqrt{10-2\sqrt{5}})$, d = $-1-\sqrt{5}-i\sqrt{2(5-\sqrt{5})}$

and

c = $1+\sqrt{5}$, d = 4

For the first solution, I knew that a and b had to be conjugates of c and d,

$a = \bar{c}$ and $b = \bar{d}$

so I evaluated $(x^2+cx+d)(x^2+\bar{c}x+\bar{d})$, and the result was my original quartic.

That meant that this factorization worked:

$(x^2+cx+d)(x^2+\bar{c}x+\bar{d}) = x^4+2x^3+4x^2+8x+16$, where

c = $\frac{1}{4}(4 - i\sqrt{50-10\sqrt{5}} - 3i\sqrt{10-2\sqrt{5}})$, d = $-1-\sqrt{5}-i\sqrt{2(5-\sqrt{5})}$

I did not pay much attention to the other, real solution to the system though, because I had already proven that the coefficients of the quadratics could not be real through my graph with no x-intercepts (see figure 2).

I then used the quadratic formula on my new quadratic factors to get the four linear factors of the quartic. The linear factors that resulted were:

$$\frac{\left(x-\frac{1}{8}(-4-i\sqrt{10(5-\sqrt{5})}-3i\sqrt{10-2\sqrt{5}})\pm\sqrt{16(4+4\sqrt{5}-4i\sqrt{2(5-\sqrt{5})})+(-4-i\sqrt{10(5-\sqrt{5})}-3i\sqrt{10-2\sqrt{5}})^2}\right)}{2}$$

and

$$\frac{\left(x-\frac{1}{8}(-4+i\sqrt{10(5-\sqrt{5})}+3i\sqrt{10-2\sqrt{5}})\pm\sqrt{16(4+4\sqrt{5}+4i\sqrt{2(5-\sqrt{5})})+(-4+i\sqrt{10(5-\sqrt{5})}+3i\sqrt{10-2\sqrt{5}})^2}\right)}{2}$$

I finally had my problem solved. I had found the four linear factors to the quartic.

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The Snag: "Did I Just Break Algebra?!"

To wrap up my problem, I went back to check my work one last time. As I was doing this, I came across those two equations again:

$$0=-c^2+2c+\frac{16}{d}+d-4$$

$$0=(2-c)d+\frac{16c}{d}-8$$

I remembered that I had graphed these equations before when I was solving over the set of real numbers, and that there had been a horizontal line that had gone across the graph.

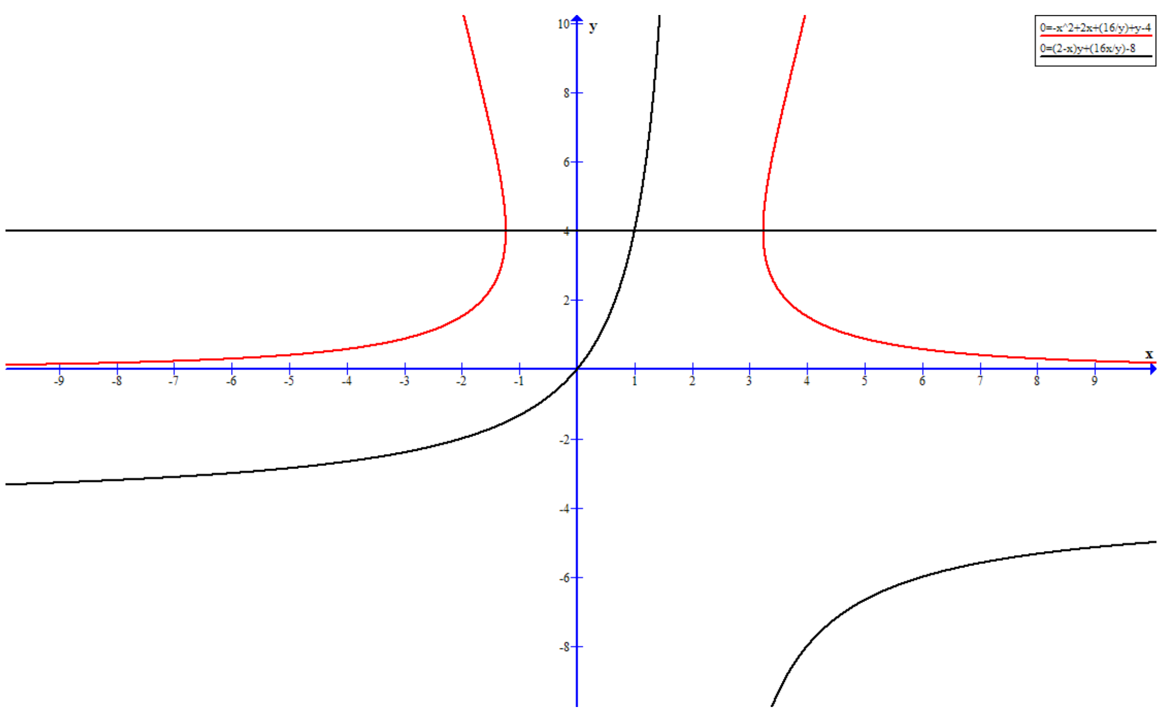


Figure 3

I had never seen this type of line in a rational function graph before, and I had assumed that it was a mistake by the program. The line was at y = 4, and seemed very out of place. If this line wasn't there, the two graphs would never intersect, and this was part of my reasoning for why the coefficients of the quadratic could not be real.

I now realized that the line was a legitimate part of the graph, and that I should not have made these assumptions before. If you look at the equation:

$$0 = (2 - x)y + (16\tfrac{x}{y}) - 8$$

You can see that when y = 4, it does not matter what the x-value is. When y = 4, the equation becomes:

$0 = 8-4x+4x-8$

which simplifies to 0 = 0. This makes a horizontal line at y = 4, because x can be anything at this point and the relation will still equal 0.

I retried the values that graph suggested as solutions, and they did work. I now had a second factorization:

$$(x^2+(1+\sqrt{5})x+4)(x^2+(1-\sqrt{5})x+4)$$

This is the point where I began to get worried that I had made a mistake in my work. It is a basic principle of algebra that a polynomial can have only one correct linear factorization, and finding this second set of quadratic factors therefore could pose a problem. The only way that both pairs of quadratics could be proper factorizations is if they both factored into the same linear factors. So, I used the quadratic formula to find the linear factors of this second pair of quadratics.

$$(x - (\frac{-1-\sqrt{5} \pm \sqrt{(1+\sqrt{5})^2 - 16}}{2}))$$

and

$$(x - (\frac{-1+\sqrt{5} \pm \sqrt{(1-\sqrt{5})^2 - 16}}{2}))$$

These did not look like the same factors that had come out of my earlier answer. But could they be a simplified version of my previous answer? I used Mathematica to test their equality, and it said "True." I couldn't have been more happy. Crisis: averted.

Wrapping It Up: The Aftermath

I had come up with two sets of quadratic factors to my quartic, but only one set of linear factors. At first, this did not seem possible, but it is possible because the factors can be grouped together to multiply into many pairs of quadratics. Here's what I mean:
Let's say the factors are:

(x-a)
(x-b)
(x-c)
(x-d)

These can be multiplied in the conventional way, like this:

(x-a)(x-b)(x-c)(x-d)

But if you group the factors in groups of two using the associative property, it would look like this:

((x-a)(x-b))*((x-c)(x-d))

You can then move the factors around in these groups to make different pairs of quadratics using the commutative property. There are a total of three combinations that you can make:

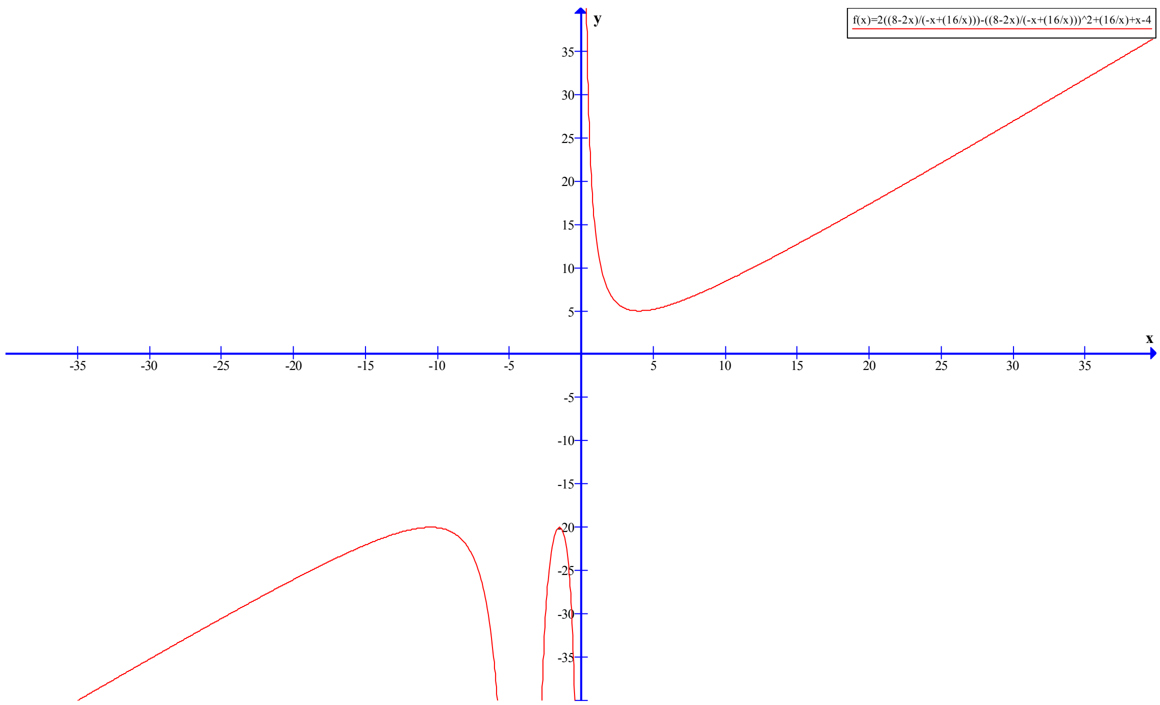
((x-a)(x-b))*((x-c)(x-d))
((x-a)(x-c))*((x-b)(x-d))
((x-a)(x-d))*((x-c)(x-b))

Using two different methods, I found two of these quadratic pairs in my earlier work. Now, I used Mathematica to find the third as well, which is the same as my first solution, but with some of the signs reversed. So the quartic factors into three pairs of quadratics, one real and two complex, but into only one set of linear factors, all complex.

The only issue I had now is that I had "proven" that there were no real quadratic factors, but I had found a pair of real quadratic factors. I had somehow made a mistake in my proof. It turns out these real values were a bit more elusive than had I realized.

Here is the equation and the graph that I had used to prove this:

$$y = \frac{16-4x}{-x+\frac{16}{x}} - \left(\frac{8-2x}{-x+\frac{16}{x}}\right)^2 + \frac{16}{x} + x - 4$$



In this equation, x is really d, because this is the variable that I had isolated. I concluded that because the graph had no zeroes, d could not be real. But in my real quadratic, d = 4. If you put four into the equation, you end up with y = $4+\frac{0}{0}$. This created a removable discontinuity at the point x = 4, my solution. 4 is not a solution to this equation, but it is a solution to the equations that I manipulated to get this one. The reason that this solution disappeared once I made this equation is because the equations I had used to make this function were relations, and this one is a function. While this solution appears as a line in a relation graph, it appears as a discontinuity in a function graph. So really, even though I though I had proven that d could not be real, it was just an issue of discontinuities and functions. d could be real after all.

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If Only I Had More Time: Extensions of the Problem

So now that I have this solution, where could I go from here? Possible extensions include:

- Finding a general method for solving quartics using systems of equations
- Exploring how changing the order of linear factors affects the higher degree factors of higher degree polynomials
- Solving this problem without the use of Mathematica
- Solving a quintic with all imaginary factors

If I had more time on this project, I would explore one of these topics. I guess they can be for my next project.

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