AoFb – Winter 08-09
The Bay School

Capstone Project Problems

1. Factoring $x^n - a^n$

   Recall that you can factor $x^2 - a^2$ ("the difference of two squares") as $x^2 - a^2 = (x-a)(x+a)$. However, you cannot factor $x^2 + a^2$ ("the sum of two squares") over the set of real numbers.

   Explore the factorizations over the set of real numbers of $x^n - a^n$ and $x^n + a^n$ for all positive integers $n \geq 3$. Describe and explain (deductively if possible) the patterns that you find.

2. Equilateral triangles with integer lattice points

   A *lattice point* is a point in the *xy*-plane with integer *x*- and *y*-coordinates. For example, (4, 1) is a lattice point, while (4, 3.7) is not.

   Is it possible for an equilateral triangle to have all three of its vertices on lattice points? If so, find one. If not, prove that it is impossible.

3. Visualizing non-real roots of quadratic functions

   Suppose a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ has two nonreal roots. How can you visualize the roots of f on its graph in the *x*-*y* plane? How and where do these points appear on the graph?

4. The shape of coffee cup sleeves

   When you buy coffee "to go", you usually get the coffee in a paper cup along with a cardboard sleeve to help you hold the cup comfortably until the liquid cools.

   

   Coffee cup sleeves are made by cutting out a piece from a flat sheet of cardboard and then wrapping the piece around the cup.

   Suppose you know the dimensions of a coffee cup. Determine the shape and size of the piece that must be cut from a flat sheet of cardboard in order to form a sleeve for the cup.

5. Analyzing Dining Plans at Walt Disney World

When people visit Walt Disney World, they often try to save money by purchasing a pre-paid meal plan rather than paying out of pocket for each meal. Disney offers three different kinds of meal plans: the Regular Dining Plan, the Quick Service Dining Plan, and the Deluxe Dining Plan.

- The Regular Dining Plan provides 1 quick service meal (think "fast food"), 1 table service meal (think "nice meal in a sit-down restaurant") without an appetizer, and 1 snack per day for $39.99.
- The Quick Service Dining Plan provides 2 quick service meals and 2 snacks per day. In addition, each person who books this package gets 1 "play free for 30 minutes in an arcade" certificate and 1 souvenir mug. The cost per day of this package is $29.99.
- The Deluxe Dining Plan provides 1 quick service meal, 2 table service meals with one appetizer per meal, and 2 snacks per day. Each person who books this plan also gets 1 souvenir mug. The cost per day of this package is $71.99.

What exact dollar value does Disney place on each of the components (quick service meal, appetizer, etc) of these three dining plans?

6. The effect of changing the value of the linear coefficient in a quadratic function

Let $f(x) = ax^2 + bx + c$ be a quadratic function. In previous courses, you learned that:

- as a increases in magnitude, the graph of f gets "skinnier"
- if $a > 0$, the graph of f is concave up, and if $a < 0$, the graph of f is concave down
- changing the value of c translates the graph vertically

What specific effect does the value of b have on the graph of f? How can you visualize the effect of this parameter on the graph of f?

7. The Ferret Chase

Suppose three ferrets are standing on the vertices of an equilateral triangle. They begin running clockwise, all at the same constant rate, chasing the ferret in front of them. At any given moment, a ferret is heading straight towards the ferret in front of him.

Do the ferrets ever catch each other? If so, when? At any time t, how can you mathematically calculate the location of a given ferret? Make a detailed, beautiful, scale drawing of the paths of the ferrets.

8. Transformational geometry, revisited

   In Math 2, you learned about some basic transformations and found the matrix representations for these transformations. However, there were plenty of transformations you did not learn about:

   - rotations through angles other than 90°, 180°, 270° and 360°
   - rotations around points other than the origin
   - reflections across lines with slopes not equal to 1 or -1
   - reflections across lines not containing the origin

   Explore and explain the workings of these other transformations verbally, graphically, symbolically, and using matrices.

9. Bus map of San Francisco

   We are used to thinking about map distances as measurements of feet, meters, etc between two points. Most maps are based on this notion of distance. However, for those of us who get around by riding the bus, a more informative map would show the distance between two points as the amount of time it takes to get from one place to the other on the bus. Points far apart would take proportionally longer to get between on the bus than points closer together.

   Create such a map of San Francisco.

10. Angles in space

    An equilateral tetrahedron is formed when four equilateral triangles are assembled into a triangular pyramid. Of course, while the angles in each equilateral triangle are 60°, most people would (correctly) disagree with the statement: "the interior angles of a tetrahedron are 60°". The same problem occurs with the other platonic solids, those 3d shapes whose faces are squares, regular pentagons, regular hexagons, etc.

    How are measures of angles measured, defined, and calculated in three-dimensions? What are the interior angles of each platonic solid?

11. Rates of change, revisited

    Over the years, we have dealt with functions which have both constant and non-constant rates of change. Even when a curve doesn't have a constant rate of change, there is still always a pattern in its changing rate of change. For example, there exists a curve $f(x)$ whose rate of change at any point (x, y) is equal to x^2.

    What is the equation for $f(x)$? How do we find equations for curves if we only know the patterns in their rates of change?

12. Create your own adventure

    There are lots of cool problems out there. The ones above are just a few that Dave and Andy came up with. Some of them were generated by questions posed by students.

    If you have a cool problem, or a cool idea that you think could turn into a problem, tell your teacher. If it works out, that problem can be one of your two project problems.